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LETTER TO THE EDITOR

# Analyticity of the partition function of the random energy model†

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**Abstract.** We consider the random energy model with complex temperature. For $\beta$ lying in a suitable set the function $Z_n(\beta)/EZ_n(\beta)$ is, at the thermodynamic limit, an analytic function of $\beta$. Analytic properties of the high temperature free energy follow directly.

Phase transitions are closely related, via the Lee–Yang theorem [11], to analytic properties of the thermodynamic functions. For random systems however, the study of such properties is not extensively developed. In this letter, we use a simple rigorous argument to study the analyticity of the partition function of the random energy model (REM) in the complex plane of temperature. This problem was investigated recently by a different approach [4] and numerically in [12]. We shall define in the following the set of the analyticity of the partition function which coincides with the high temperature region.

The REM, introduced in [3] as a simple tractable model sharing some properties of a spin glass, has been investigated in the last few years using different approaches [13, 6, 5]. In its original formulation, the REM describes a system of $d^n$, $(d \geqslant 2)$, independent, identically distributed energy levels $\varepsilon_x$ and the partition function is expressed as the statistical sum

$$Z_n(\beta) = \sum_{x=1}^{d^n} \exp(-\beta\varepsilon_x).$$

In the original formulation it is assumed that the variables $\varepsilon_x$ are independent (for different $x$'s) and distributed according to a Gaussian law of zero mean, $E\varepsilon_x = 0$, and variance depending on the generation $n$, $E\varepsilon_x^2 = n$, for every $x$ and $E\varepsilon_i^2 = n$ [3] ($E(\cdot)$ denotes the expectation). A mathematical reformulation of the model is given by Ruelle [13] in terms of Poisson distributions. This idea was also explored in [6, 10, 5].

The study of the model is made possible by a reformulation allowing expression of the variables $\varepsilon_x$ in terms of sums of random variables with the same law. Namely, by noticing that a Gaussian variable is uniquely characterized by its mean and its variance, one can consider the family

$$\{\xi_k(x), k = 1, \ldots, n, x = (x_1, \ldots, x_n) \in X_n = \{1, \ldots, d\}^n\}$$

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of independent Gaussian variables, having, for every $k$, common distribution with the Gaussian variable $\xi$ with $E\xi=0$ and $E\xi^2=1$. In this reformulation, the variable $\varepsilon_x$ representing the energy level at the $n$th generation is expressed as the sum $\varepsilon_x=\Sigma_{k=1}^n\xi_k(x)$. This equivalent representation has the advantage of having the same law for all the random variables used in the model, contrary to the original formulation where the probability distribution changes from level to level.

We assume in the following that the inverse temperature $\beta$ can take complex values $\beta=\beta_1+i\beta_2$ and the analytic properties of the partition function in a domain of the complex $\beta$ plane will be examined. One can easily see that the partition function given by

$$Z_n(\beta)=\sum_{x\in X_n}\exp\left(-\beta\sum_{k=1}^{n}\xi_k(x)\right)$$

represents the same random variable as the REMs partition function. We have, moreover, that

$$Z_{n+1}(\beta)=\sum_{x\in X_{n+1}}\prod_{k=1}^{n+1}\exp(-\beta\xi_k(x))=\frac{1}{d}\sum_{j=1}^{d}\prod_{k=1}^{n}\exp(-\beta\xi_k(x))Z_1^{(j)}(\beta)$$

where $Z_1^{(j)}(\beta)$ is a random variable having the same law as the first generation partition function $Z_1(\beta)$ and it is independent of the previous product.

*Proposition*. Define $M_n(\beta)\equiv Z_n(\beta)/EZ_n(\beta)$ and let $A$ be the open set

$$A=\bigcup_{1<h\leqslant 2}\operatorname{int}\left\{\beta\in C: h(h-1)\frac{\beta_1^2}{2}+h\frac{\beta_2^2}{2}<(h-1)\log d\right\}.$$

Then, $M_n(\beta)$ converges as $n\to\infty$, uniformly for $\beta\in A$, and in the mean (with respect to the distribution of $\xi$s to the finite limit $M_\infty(\beta)$.

Before giving the proof of the proposition, let us remark that from the definition of the process $M_n(\beta)$, it follows that

$$E(M_{n+1}(\beta)|\mathcal{F}_n)=M_n(\beta)$$

i.e. $M_n(\beta)$ is a martingale with respect to the $\sigma$-algebra $\mathcal{F}_n$ containing information about the first $n$ generations. For real $\beta$ this kind of non-negative martingales, arising in different mean field models, have been investigated in detail in [2, 6–9]. In particular, conditions for their almost sure, and in the mean convergence, define the critical temperature and characterize the high temperature behaviour.

On the other hand, the martingale difference $M_{n+1}(\beta)-M_n(\beta)$ gives

$$M_{n+1}(\beta)-M_n(\beta)=\sum_{x\in X_n}\frac{\Pi_{k=1}^n\exp(-\beta\xi_k(x))}{d^nE(\exp(-\beta\xi))^n}(M_1^i(\beta)-1)$$

with $E(M_{n+1}(\beta)-M_n(\beta))=0$, $M_1^i(\beta)$ being a random variable having the same distribution as $M_1(\beta)$.

The proof of the proposition uses the following lemma proved in [1].

*Lemma.* If $\{Y_i\}$ are independent complex random variables with $E(Y_i)=0$, or martingale differences, then, for $1 \leqslant h \leqslant 2$,

$$E\left|\sum Y_i\right|^h \leqslant 2^h \sum E|Y_i|^h.$$

*Proof of the proposition.* As $\beta$ is a complex variable, we have to show that there exists a variable $M_\infty(\beta)$ such that

$$\sup_{\beta \in A} |M_n(\beta) - M_\infty(\beta)|$$

converges to zero almost surely and in the $h^{\text{th}}$ mean as $n \to \infty$. Let $B(\beta^*, r)$ be the disc of centre $\beta^* \in A$ and radius $r$ and $\partial B(\beta^*, r)$ its boundary i.e. $\{\beta' : |\beta^* - \beta'| = r\}$. It is sufficient to prove uniform convergence in $B(\beta^*, r)$ for each $\beta^* \in A$.

We are given the following parameterization of $\partial B(\beta^*, 2r)$

$$\partial B(\beta^*, 2r) = \{\psi(t) : \psi(t) = \beta^* + 2re^{2\pi i t}, t \in [0, 1)\}.$$

Using the maximum-modulus principle for an analytic function $\phi(\beta)$ on $B(\beta^*, 3r)$, we have that

$$\sup_{\beta \in B(\beta^*, r)} |\phi(\beta)| \leqslant 2 \int_0^1 |\phi(\psi(t))| \mathrm{d}t.$$

Remarking now that the difference $M_m(\beta) - M_n(\beta)$, $m \geqslant n$, is analytic on $A$, and applying the previous bound we obtain

$$\sup_{\beta \in B(\beta^*, r)} |M_m(\beta) - M_n(\beta)| \leqslant 2 \int_0^1 \sum_{n=0}^{\infty} |M_{n+1}(\psi(t)) - M_n(\psi(t))| \mathrm{d}t.$$

On the other hand, one can check that the function $|M_{n+1}(\psi(t)) - M_n(\psi(t))|$ is uniformly integrable since its expectation is finite: indeed

$$\int_0^1 \sum_{n=0}^{\infty} E|M_{n+1}(\psi(t)) - M_n(\psi(t))| \, \mathrm{d}t \leqslant \sup_{\beta \in \partial B(\beta^*, 2r)} \sum_{n=0}^{\infty} E|M_{n+1}(\beta) - M_n(\beta)|.$$

Now, applying the lemma to the previously given expression for the difference $M_{n+1}(\beta) - M_n(\beta)$ we obtain

$$\sum_{n=0}^{\infty} E|M_{n+1}(\beta) - M_n(\beta)| \leqslant \sum_{n=0}^{\infty} \left( \frac{e^{2\beta_1^2 n}}{d^{2n} e^{(\beta_1^2 - \beta_2^2)n}} E|M_1'(\beta) - 1|^2 \right)^{1/2}$$

$$\leqslant \text{const} \sum_{n=0}^{\infty} \left( \frac{e^{\beta_1^2}}{d e^{(\beta_1^2 - \beta_2^2)/2}} \right)^n.$$

Now, if $(\beta_1^2)/2 + (\beta_2^2)/2 < \log d$, $\sup_{\beta \in A} |M_n(\beta) - M_\infty(\beta)|$ converges almost surely and in the mean as $n \to \infty$. The proof is complete by remarking that the set $A$ can be covered by open discs centred at each $\beta \in A$.

As a consequence, we have the

*Corollary.* $M_\infty(\beta)$ is analytic on $A$.

The open set $A$ contains the high temperature region of the real axis. One can also study the specific free energy of the model. From the previous result, it follows immediately that the specific free energy is an analytic function of $\beta$, for $\beta \in A$, and that annealed and quenched free energies coincide.

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